Lecture 11: Projective Morphisms

Note Title 10/4/2019

Recall: $\mathbb{P}^n_A := \operatorname{Proj} \underbrace{A[x_0, \cdots, x_n]}_{S}$

$\mathcal{O}(1) = \widetilde{S(1)}$ invertible sheaf

generated by global sections $x_0, \cdots, x_n$

Theorem:
$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr] & \mathbb{P}^n_A \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd} \approx$$
$\mathcal{L}$: invertible sheaf on X generated by global sections $\boxed{\begin{matrix} s_0, \cdots, & s_n \\ \| & \| \\ \varphi^*(x_0) & \varphi^*(x_n) \end{matrix}} \in \Gamma(X, \mathcal{L})$

gives a framing

pf: $(\Longrightarrow)$ $\mathcal{L} = \varphi^*(\mathcal{O}(1))$, $s_i = \varphi^*(x_i)$

$(\Longleftarrow)$ $X_i := \{ x \in X \mid s_x \notin m_x \} \underset{\text{open}}{\subseteq} X$

$\mathcal{L}$ generated by global sections $s_i \Longrightarrow \bigcup_{i=0}^n X_i = X$

$A\left[\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right] \longrightarrow \Gamma(X_i, \mathcal{O}_X) \quad \approx \quad X_i \longrightarrow U_i \subseteq \mathbb{P}^n_A$

$\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i}$ $\mathcal{L}$ invertible

glue to $X \to \mathbb{P}^n_A$

ex. (Automorphism of $\mathbb{P}^n_k$)

$A = (a_{ij}) \in GL(n+1, k)$ induces an automorphism of $\mathbb{P}^n_k$

such action coincides w/ the one induced by $\underset{\underset{k^*}{\Uparrow}}{\lambda} A$

$PGL(n, k) := GL(n+1, k) / k^*$

need $\underset{(n+1)^2 - 1/n}{\overset{}{(n+2)}}$ points to determine

$(n+2) = (n+1)^2 - 1/n$

Conversely, $\varphi \in \mathrm{Aut}(\mathbb{P}^n_k) \rightsquigarrow \underset{\mathbb{Z}}{\underset{\cong}{\mathrm{Pic}(\mathbb{P}^n_k)}} \xrightarrow[\cong]{\varphi^*} \underset{\mathbb{Z}}{\underset{\cong}{\mathrm{Pic}(\mathbb{P}^n_k)}}$

generated by $\mathcal{O}(1)$

φ^* sends generator to generator

$\mathcal{O}(1)$ has global section while $\mathcal{O}(-1)$ has none

Thus, $\varphi^*\mathcal{O}(1) = \mathcal{O}(1)$ & $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1)) \xrightarrow{\varphi^*} \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$

$\cong$ linear functions in $x_0, \cdots, x_n$ as k-modules

$\therefore \varphi^* \in GL(n+1, k)$

Q: How about $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$?

Proposition 1: $\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{P}^n_A \\ & \searrow \ \swarrow & \\ & \mathrm{Spec} A & \end{array}$ $\quad \mathcal{L} := \varphi^*\mathcal{O}(1)$, $s_i = \varphi^*(x_i) \in \Gamma(X, \mathcal{L})$

φ: closed embedding $\iff$
- $X_i \subseteq X$ affine
- $A\left[\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right] \twoheadrightarrow \Gamma(X_i, \mathcal{O}_X)$

i.e X projective over A

pf: $(\Rightarrow)$ $X_i = X \cap \underset{\text{affine}}{\underline{U_i}}$ $\therefore X_i$ closed subscheme of U_i

$(\Leftarrow)$ $X_i = \varphi^{-1}(U_i)$ closed subscheme of U_i

& covers X

Proposition 2: k = algebraically closed, X projective $/k$

$\mathbb{P}^N_k \leftarrow X \xrightarrow{\varphi} \mathbb{P}^n_k$, $\mathcal{L} = \varphi^*\mathcal{O}(1)$, $s_i = \varphi^*(x_i) \in \Gamma(X, \mathcal{L})$

$V = \mathrm{Span}\langle s_i \rangle_k \subseteq \Gamma(X, \mathcal{L})$

will $\neq$ hold ?

φ closed immersion $\iff$ • V separates points

$\forall\, x_1, x_2 \in X$ closed points

$\exists\, s \in V$ s.t. $s_{x_1} \in m_{x_1}\mathcal{L}_{x_1}$, $s_{x_2} \notin m_{x_2}\mathcal{L}_{x_2}$

• V separates tangents

$\{ s \in V \mid s_x \in m_x \mathcal{L}_x \}$ spans $m_x\mathcal{L}_x / m_x^2 \mathcal{L}_x$, $\forall x \in X$

pf: ($\Rightarrow$)

• $\varphi(x_1), \varphi(x_2) \in \mathbb{P}^n_k$, $\exists$ hyperplane $\sum a_i x_i = 0$ passing through $\varphi(x_1)$ but not $\varphi(x_2)$, can take $s = \sum a_i s_i$

image of closed points are closed

Here we inexplicitly use every closed point is $(b_0, \cdots, b_n) \in \mathbb{P}^n_k$

• Up to coordinate change, may assume $x = [1:0:\cdots:0]$

Identify $x = (0,\cdots,0) \in U_0 = \operatorname{Spec}\left[\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right]$

m_x/m_x^2 is spanned by $\frac{x_i}{x_0}$

$\frac{x_i}{x_0} \mapsto s_i = \varphi^*(x_i)$ vanishes at x.

$m_x/m_x^2 \cong m_x\mathcal{L}_x / m_x^2\mathcal{L}_x$

($\Leftarrow$) V separates points $\Rightarrow$ $\varphi: X \to \mathbb{P}^n_k$ injective on closed points

thus injective on all points

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\substack{\text{projective} \\ \Downarrow \\ \text{proper}}"'] & & \mathbb{P}^n_k \arrow[dl, "\substack{\text{projective} \\ \Downarrow \\ \text{separated}}"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

$\Rightarrow$ φ is proper. In particular, $\varphi(X)$ closed

To show $\mathcal{O}_{\mathbb{P}^n_k} \twoheadrightarrow \varphi_*\mathcal{O}_X$, it suffices to check

$\mathcal{O}_{\varphi(x), \mathbb{P}^n_k} \twoheadrightarrow (\varphi_*\mathcal{O}_X)_x \cong \mathcal{O}_{x,X}$, $\forall x \in X$

Coherent $\mathcal{O}_{\mathbb{P}^n_k}$-module

enough to check on closed points.

$A \to B$ surjective $\Leftrightarrow$ $A_{\mathfrak{p}} \twoheadrightarrow B_{\mathfrak{p}}$ $\forall\, \mathfrak{p} \in \operatorname{Spec} B$ $\Leftrightarrow$ $A_{\mathfrak{m}} \twoheadrightarrow B_{\mathfrak{m}}$ $\forall\, \mathfrak{m} \triangleleft B$ maximal

Both residue fields of $\mathcal{O}_{\varphi(x), \mathbb{P}^n_{\mathbb{k}}}$ & $\mathcal{O}_{x,X}$ are $\mathbb{k}$

V separates tangents $\Longrightarrow$ $\mathfrak{m}_{\varphi(x), \mathbb{P}^n_{\mathbb{k}}} \twoheadrightarrow \mathfrak{m}_{x,X} / \mathfrak{m}_{x,X}^2$

Lemma 1: $f: A \to B$ local homomorphism of local Noetherian rings

s.t. ① $A/\mathfrak{m}_A \xrightarrow{\simeq} B/\mathfrak{m}_B$

② $\mathfrak{m}_A \twoheadrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ $\Longrightarrow$ $f: A \twoheadrightarrow B$

③ B finitely generated A-module

pf: (Nakayama's lemma)

A: local ring, M: finitely generated A-module

then $m_1, \cdots, m_k$ generate M as A-module

iff $m_1, \cdots, m_k$ generate $M \otimes_A A/\mathfrak{m}_A$ as $A/\mathfrak{m}_A$-module

- $\mathfrak{a} = \mathfrak{m}_A B \subseteq \mathfrak{m}_B$ viewed as B-module

  $\mathfrak{a} \otimes_B B/\mathfrak{m}_B \twoheadrightarrow \mathfrak{m}_B \otimes_B B/\mathfrak{m}_B \cong \mathfrak{m}_B/\mathfrak{m}_B^2$

  $\xrightarrow[\text{Nakaya's lemma}]{} \mathfrak{a} = \mathfrak{m}_B$

- $A \twoheadrightarrow B \Leftrightarrow B$ is generated by $\underset{B}{\overset{\in}{1}}$ as A-module

  $1 \in A/\mathfrak{m}_A$ generator $\xrightarrow{\cong} B/\mathfrak{m}_B \xleftarrow{\cong} B/\mathfrak{m}_A B$

  B finitely generated as A-module $\xrightarrow[\text{Nakayama's lemma}]{}$ $\overset{1_B}{\overset{\parallel}{f(1_A)}}$ generate B

Definition: X: Noetherian scheme, $\mathcal{L}$: invertible sheaf

$\mathcal{L}$ is <u>ample</u> if for every coherent sheaf $\mathcal{F}$ on X

$\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global section, $n \geq n_0(\mathcal{F})$

ex. Every invertible sheaf on an affine scheme is ample

ex. (Serre) Every very ample invertible sheaf is ample

$p \in E$ elliptic curve, generic $\quad \mathcal{O}_E(p)$ is ample but NOT very ample

Remark: • Ample is an intrinsic notion compare to <u>very ample</u>.

via an embedding

• $\mathcal{L}$: ample$/_{\mathbb{C}}$ $\Longleftrightarrow$ $\exists$ positive metric on the corresponding line bundle.

Proposition 3: With above notation

TFAE ① $\mathcal{L}$ ample

② $\mathcal{L}^{\otimes n}$ ample for all $n > 0$

③ $\mathcal{L}^{\otimes n}$ ample for some $n > 0$

obvious from metric point of view

pf: ① $\Rightarrow$ ② $\Rightarrow$ ③ trivial

③ $\Rightarrow$ ① $\quad \forall\, \mathcal{F}$ coherent sheaf

$\mathcal{F} \otimes (\mathcal{L}^{\otimes n})^m$ generated by global section for $m > m_0$

$(\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \otimes (\mathcal{L}^{\otimes n})^m \quad = \quad m > m_k$

$k = 1, \cdots, n$

Then $\mathcal{F} \otimes \mathcal{L}^{m > n \cdot \max\{m_F\}} = \left(\mathcal{F} \otimes \mathcal{L}^{\otimes k}\right) \otimes \left(\mathcal{L}^{\otimes n}\right)^{l > m_x}$

is generated by global sections

Theorem: $X =$ scheme of finite type over Noetherian ring A

$\mathcal{L} =$ invertible sheaf on X

Then $\mathcal{L}$ is ample iff $\exists n > 0$, $\mathcal{L}^{\otimes n}$ very ample

pf: ($\Leftarrow$) $\mathcal{L}^{\otimes n}$ very ample i.e. $\exists\, i : X \hookrightarrow \mathbb{P}^n_A$

s.t. $\mathcal{L}^{\otimes n} = i^* \mathcal{O}(1)$

$\bar{X} := \overline{i(X)}$ is projective. $\bar{i}^* \mathcal{O}(1)$

$\bar{X} \xrightarrow{\bar{i}} \mathbb{P}^n_A$

Every coherent sheaf $\mathcal{F}$ on $X \rightsquigarrow \bar{\mathcal{F}}$ on $\bar{X}$ coherent

w/ $\bar{\mathcal{F}}|_X = \mathcal{F}$

then Serre's theorem $\bar{\mathcal{F}} \otimes \left(\bar{i}^* \mathcal{O}(1)\right)^n$ generated by global sections

preserved by pull-back (to X)

($\Rightarrow$) $\mathcal{L}$ ample

$\forall p \in X$, $U \ni p$ affine open set in X s.t. $\mathcal{L}|_U \cong \mathcal{O}_U$

$Y = X \setminus U$ closed $\rightsquigarrow$ $\mathcal{I}_Y \subseteq \mathcal{O}_X$

w/ reduced structure   coherent

$\mathcal{L}$ ample $\Longrightarrow$ $\mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}$ generated by global sections

$n \gg 0$

$\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^{\otimes n}) \subseteq \Gamma(X, \mathcal{L}^{\otimes n})$

In particular, $\exists s \in$ s.t. $s_p \notin m_p \mathcal{L}_p$

$$p \in X_s = \{ x \in X \mid s_x \notin m_x \mathcal{L}_x \} \subseteq U \underset{\text{affine}}{\cong} \operatorname{Spec} A$$

$s \in \Gamma(X, I_Y \otimes \mathcal{L}^n) \quad \therefore \quad X_s \cap Y = \phi$

$$\mathcal{L}|_U \cong \mathcal{O}_U \cong A \qquad X_s \cong \operatorname{Spec}(A_f) \text{ affine}$$

$$s \longmapsto f$$

To sum up, for every $p \in X$

$$\exists\, s \in \Gamma(X, \mathcal{L}^{\otimes n_p}) \text{ s.t. } \underset{p}{X_s} \text{ affine}$$

X Noetherian $\Longrightarrow$ $\exists$ finite open affine cover $\{X_{s_i}\}$

$$s_i \in \Gamma(X, \mathcal{L}^{\otimes n_i}) \qquad n = \prod n_i \qquad X_{s^k} = X_s$$

It suffices to show that $\mathcal{L}^{\otimes n}$ is very ample.

$$X_{s_i} \cong \operatorname{Spec} B_i \Longrightarrow B_i \cong A[b_{ij}]/\sim, \quad b_{ij} \in \Gamma(X_{s_i}, \mathcal{O}_X)$$

X of finite type

$$\exists\, n \in \mathbb{N} \text{ st } s_i^n b_{ij} \in \Gamma(X, \mathcal{L}^{\otimes n})$$

The sections $\underset{i=1,\dots,k}{s_i}$, $\underset{i=1,\dots,k,\ j=1,\dots,k_i}{s_i^n b_{ij}} \in \Gamma(X, \mathcal{L}^{\otimes n})$

defines a morphism $X \xrightarrow{\varphi} \mathbb{P}^N_A$ coordinate x_i, x_{ij}

actually s_i already defines the morphism $\because \bigcup_i X_{s_i} = X$

Claim: φ is an immersion

$$X_{s_i} \longrightarrow U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n_A$$

$$B_i \twoheadleftarrow A\left[\frac{x_j}{x_i}, \frac{x_{kl}}{x_i}\right]$$

$$b_{ij} \longleftarrow \frac{x_{ij}}{x_i^n}$$

$X_{s_i} \hookrightarrow U_i$ as closed subscheme

$$X = \bigcup X_{s_i} \longrightarrow \bigcup_i U_i \underset{\text{open}}{\subseteq} \mathbb{P}^n_A$$

ex. $X = \mathbb{P}^n_k$, $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}(1) \rangle$ Fubini–Study metric

$\mathcal{L}$ is ample iff $\mathcal{L} \cong \mathcal{O}(d)$, $d > 0$

$\mathcal{O}(d)$ is very ample $\mathbb{P}^n_k \longrightarrow \mathbb{P}_k^{\binom{d+n}{n}-1}$

$x_j \longmapsto x_0^{i_0} \cdots x_n^{i_n}$, $\sum_j i_j = d$